

INTRINSICALLY LINKED GRAPHS WITH KNOTTED COMPONENTS

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ABSTRACT. We construct a graph G such that any embedding of G into R^3 contains a nonsplit link of two components, where at least one of the components is a nontrivial knot. Further, for any $m < n$ we produce a graph H so that every embedding of H contains a nonsplit n component link, where at least m of the components are nontrivial knots.

We then turn our attention to complete graphs and show that for any given n , every embedding of a large enough complete graph contains a two component link whose linking number is a nonzero multiple of n .

1. INTRODUCTION

We call a graph *intrinsically linked* (*intrinsically knotted*) if every embedding of that graph into three-space contains a nontrivial link (knot). Intrinsically linked and intrinsically knotted graphs were first studied in the early 1980s by Sachs [9] and by Conway and Gordon [1]. Since then these graphs have been extensively studied. Significant progress has been made in this area, such as the complete classification of minor minimal examples for intrinsically linked graphs by Robertson, Seymour, and Thomas [8]. Since this classification was completed, work has turned to finding graphs whose every embedding contains more complex structures. For example, one could require that every embedding of the graph contain a nonsplit link of n components [2], or a link with linking number larger than some constant [3, 10].

We would now like to combine the two properties of intrinsic linking and intrinsic knotting and ask the following question: Does there exist a graph such that any embedding of this graph contains a nonsplit link, where at least one of the components of this link is a nontrivial knot?

The answer is affirmative, and we demonstrate examples.

Theorem 2.1 *Every embedding of the graph $F(126)$ contains a nonsplit link of two components, where at least one of these two components is a nontrivial knot.*

This graph contains one hundred and twenty-six vertices, so it is unlikely to be minor minimal. In fact, it is possible that graphs as small as K_{10} or K_{11} have this property, but current techniques do not allow us to detect it.

A graph containing an n -component link in every embedding was first demonstrated by Flapan, Foisy, Naimi, and Pommersheim [2]. The link constructed in [2] has the form of a chain of components, each linked to its neighbors. A different type of n -component link, more like a ring of keys where one component links all the others, was found in [4]. Using this later construction, we can produce a link of arbitrarily many components, some of which are nontrivial knots.

Theorem 2.3 *Given $m \leq n$, there exists a graph G such that every embedding of G contains a nonsplit link of $n+1$ components, where at least m of the components are nontrivial knots.*

Flapan [3] and Shirai and Taniyama [10] proved that for a given k , certain graphs always contain a two component link with linking number greater than or equal to k . This led the author and Diesl to study graphs that contained two component links with linking number equal to a multiple of k , for $k = 2^r$ [4]. Here we continue this work and search for links with linking number equal to a multiple of a given integer n .

Let $\eta_n = \alpha'_n(\zeta_n + 3)$ where $\zeta_n = (n+1)(\lfloor \frac{n+1}{2} \rfloor)(\lceil \frac{n+1}{2} \rceil)^{\lceil \log_2(n) \rceil - 2}$, and α'_n is the sequence from [4] defined as $\alpha'_1 = 1$, $\alpha'_{2m} = 2\alpha'_{2m-1}$, and $\alpha'_{2m+1} = 2\alpha'_{2m} + 1$. Then we have the following theorem.

Theorem 3.1 *Given $n \geq 5$, every embedding of K_{η_n} contains a 2-component link L, Z with $lk(L, Z) = kn$ for some $k \neq 0$.*

The author and Diesl produced the analogous result for the case of $n = 2^r$ and $n = 3$ in [4]. When $n = 2^r$ the number of vertices required for the graph in Theorem 3.1 is roughly $\alpha'_n 2^{r^2}$ whereas the bound from [4] is smaller, roughly $\alpha'_n 2^{\frac{r^2}{2}}$. So, while here we are able to extend the construction to all n , this extension comes at a cost.

2. KNOTTED LINKS

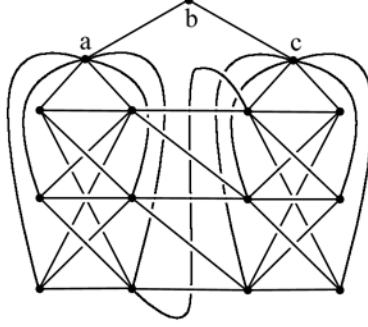
To prove Theorem 2.1, we first produce an intrinsically knotted graph F and show that a certain pair of edges in this graph are contained in a knotted cycle in any embedding. This graph F is closely related to the graph introduced by Foisy in [6].

We then take ten copies of F and carefully glue them together to produce $F(126)$. The crucial idea is that any embedding of $F(126)$ can be contracted down to a copy of K_6 , where every triangle containing a distinguished vertex is the contraction of a knotted cycle. Thus the linked triangles in K_6 correspond to a pair of linked cycles in $F(126)$ and as one of these triangles contains the distinguished vertex, the corresponding cycle is knotted.

Theorem 2.1. *Every embedding of the graph $F(126)$ contains a nonsplit link of two components, where at least one of these two components is a nontrivial knot.*

Proof. Let F be the graph shown in Figure 1. Note that this graph contains the graph produced by Foisy in [6] as a minor, and hence is intrinsically knotted.

We, however, want the edges $a - b$ and $b - c$ to always be contained in a knotted cycle, whatever the embedding of F . This graph contains two disjoint copies of $K_{3,3,1}$ which is intrinsically linked, and in fact each copy always contains a triangle linked with a square. Using the edges running between the two copies of $K_{3,3,1}$, we may contract F to the graph shown in Figure 2. We contract the triangle containing a to cycle 1, and the triangle containing c to cycle 2. Because of the edges in the middle of the graph, the triangle containing a is adjacent to the square linked with the triangle containing c . Contract this square to cycle 2, and the analogous square in the other copy of $K_{3,3,1}$ to cycle 3. This gives us that cycle 1 is linked with cycle 3, and cycle 4 is linked with cycle 2. It is shown in [5] and [11] that the graph in Figure 2 always contains a knotted cycle, and clearly this knotted cycle must use the edges $a - b$ and $b - c$. Since vertex expansion (and edge addition) do not change

FIGURE 1. The graph F .

the isotopy type of a cycle, every embedding of F contains a knotted cycle that uses the edges $a - b$ and $b - c$. (Note that this would not be true if we used the corresponding expansion of Foisy's graph from [6]).

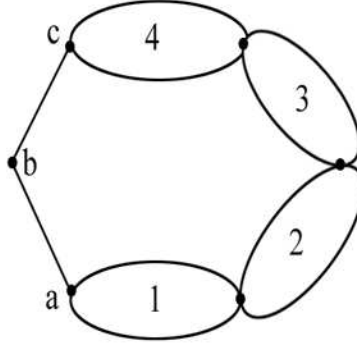


FIGURE 2. A contraction of F . The cycles labeled 1 and 4 come from the triangles containing a and c , respectively. Cycles 2 and 3 are the squares linked with the triangles corresponding to 4 and 1 respectively.

We now take ten copies of F . Label the upper vertices of the first graph a_1, b_1, c_1 and the upper vertices of the second graph a_2, b_2, c_2 and so on.

Take the first four copies and identify all the edges $a_i - b_i$. Note that we have six labeled vertices after this identification. The resulting graph appears in Figure 3. The circles containing the letter F denote the undrawn remainder of the graph F .

Take the remaining six copies of F and glue them on as follows. Identify $a_5 - b_5 - c_5$ to $c_1 - b - c_2$. Identify $a_6 - b_6 - c_6$ to $c_1 - b - c_3$. Identify $a_7 - b_7 - c_7$ to $c_1 - b - c_4$. Identify $a_8 - b_8 - c_8$ to $c_2 - b - c_3$. Identify $a_9 - b_9 - c_9$ to $c_2 - b - c_4$. Finally, identify $a_{10} - b_{10} - c_{10}$ to $c_3 - b - c_4$.

This graph is $F(126)$.

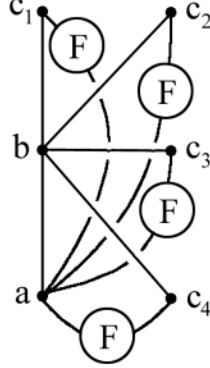


FIGURE 3. Four copies of F glued together. The edge labeled $a-b$ is the result of identifying the $a_i - b_i$.

Choose an embedding of $F(126)$. In each copy of F , the edges $a-b$ and $b-c$ are used in a knotted cycle. Contract each copy of F along this knotted cycle, deleting any edges that become parallel to it. The result is shown in Figure 4. Note that we now have a copy of K_6 such that every triangle that includes vertex b is the contraction of knotted cycle, and hence knotted.

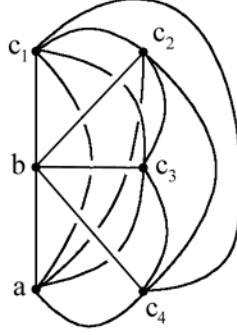


FIGURE 4. The contraction to K_6 . Note that any triangle that uses two straight lines and one curved one is knotted.

Every embedding of K_6 contains a pair of linked triangles, and one of these triangles must include the vertex b . Thus in this embedding of K_6 we have a two component link, where the component using vertex b is knotted. Again, as vertex expansion and edge addition do not change the isotopy class of these cycles, we have that every embedding of $F(126)$ contains two linked cycles, where at least one of them is a nontrivial knot. This completes the proof. \square

Using the techniques of Theorem 2.1, we can produce a slightly smaller graph with the same property as $F(126)$. We will follow the proof of Theorem 2.1 exactly, except we now glue nine copies of F together to produce a graph that contracts to $K_{3,3,1}$. A similar argument may be possible for other graphs in the Peterson family.

Proposition 2.2. *Every embedding of $F(115)$ contains a nonsplit two component link, where at least one of these components is a nontrivial knot.*

Proof. We will glue nine copies of F together to produce a graph with one hundred and fifteen vertices that contracts down to $K_{3,3,1}$.

To construct $F(115)$, take nine copies of F , labeling their vertices as before. Identify b_1, b_2, b_3 and call the resulting vertex B . We now have seven preferred vertices, $a_1, a_2, a_3, B, c_1, c_2, c_3$. These will form the $K_{3,3,1}$.

Now identify $a_4 - b_4 - c_4$ to $a_1 - B - c_2$. Identify $a_5 - b_5 - c_5$ to $a_1 - B - c_3$. Identify $a_6 - b_6 - c_6$ to $a_2 - B - c_1$. Identify $a_7 - b_7 - c_7$ to $a_2 - B - c_3$. Identify $a_8 - b_8 - c_8$ to $a_3 - B - c_1$. Identify $a_9 - b_9 - c_9$ to $a_3 - B - c_2$.

We proceed as above, contracting along knotted cycles to obtain a copy of $K_{3,3,1}$ where every triangle containing B is a knotted cycle. Every embedding of $K_{3,3,1}$ contains a triangle linked with a square, and the triangle must contain B . This gives the desired result. \square

By combining many copies of these graphs, we can construct a graph whose every embedding contains a nonsplit link with arbitrarily many knotted components. In fact, our construction produces an $n + 1$ -component link where all but one of the components are knots.

Theorem 2.3. *Given $m \leq n$, there exists a graph G such that every embedding of G contains a nonsplit link of $n + 1$ components, where at least m of the components are nontrivial knots.*

Proof. By Theorem 2.1, every embedding of the graph $F(126)$ contains a two component link L, K where K is a nontrivial knot and $lk(L, K) = 1 \pmod{2}$.

We may now apply Corollary 2.4 of [4], which states that if every embedding of H contains a two component link L, Z with non-zero linking number, then every embedding of $*^{\alpha'_n}H$ contains an $n + 1$ component link L, Z_i with $lk(L, Z_i) \neq 0$. Here $\alpha'_1 = 1$, $\alpha'_{2m} = 2\alpha'_{2m-1}$, and $\alpha'_{2m+1} = 2\alpha'_{2m} + 1$.

In the proof of this corollary, the desired link in an embedding of $*^{\alpha'_n}H$ is produced by inductively modifying cycles in pairs of links with $k - 1$ components to produce a link with k components. However, it is only necessary to modify the central component L at each stage, so the Z_i remain unchanged throughout the process.

Choose $H = F(126)$. In $G = *^{\alpha'_n}F(126)$, there are α'_n copies of $F(126)$. In the i th copy of $F(126)$, we can find a link L_i, K_i by Theorem 2.1. Thus, in $G = *^{\alpha'_n}F(126)$, we can find a nonsplit $n + 1$ component link L, Z_i , where the Z_i are the cycles K_i and the cycle L is obtained from the L_i . The K_i are nontrivial knots, so we are done. \square

3. LINKING MODULO N

In [4] the author and Diesl were able to show that for large enough m , every embedding of K_m contains a two component link with linking number $k2^r$ ($k \neq 0$). Here we extend this construction from powers of two to all n .

Crucial to these constructions is the “ring of keys” lemma (Lemma 2.2 of [4]). In any embedding of a large enough complete graph, this lemma ensures that we can

find an $n + 1$ component link L, Z_i where the preferred component L has non-zero linking number with the other n components. Further, as discussed in the proof of Theorem 2.5 of [4], using a result of Johnson and Johnson [7], this link may be constructed so that the number of vertices in each of the Z_i is bounded from below.

Let $\alpha'_{2m-1} = \frac{4^m-1}{3}$, and $\alpha'_{2m} = 2\alpha'_{2m-1}$, then Corollary 2.4 of [4] implies that if G is intrinsically linked, we may use Lemma 2.2 [4] to show every embedding of $*^{\alpha'_{n+1}}G$ contains an $n+1$ component link of the desired type. Some of the properties of the components Z_i are inherited from G ; it is this fact that is useful in the proof of Theorem 2.3.

For $n \geq 5$, let $\eta_n = \alpha'_n(\zeta_n + 3)$ where $\zeta_n = (n+1)(\lfloor \frac{n+1}{2} \rfloor)(\lceil \frac{n+1}{2} \rceil)^{\lceil \log_2(n) \rceil - 2}$. This ζ_n is the bound on the minimal number of vertices in the link components Z_i .

For $n = 2, 3, 4$, the theorem below is true and the argument proceeds with fewer vertices required in K_{η_n} . However, the bounds produced by this result for $n = 2, 3, 4$ are inferior to those of [4].

Theorem 3.1. *Given $n \geq 5$, every embedding of K_{η_n} contains a 2-component link L, Z with $lk(L, Z) = kn$ for some $k \neq 0$.*

Proof. Given n , by Lemma 2.2 of [4], in any embedding of K_{η_n} there exist disjoint cycles $L, Z_1, Z_2 \dots Z_n$ with $lk(L, Z_i) \neq 0$ and each Z_i containing at least ζ_n vertices. Choose the orientations of Z_i so that this linking number is positive. Examine the sums $S_j := \sum_{i=1}^j [Z_i]$ in $H_1(\mathbb{R}^3 \setminus L)$. As there are n such sums, either two are equal modulo n , or one is zero modulo n . In the latter case this sum is kn for $k \neq 0$ since the $[Z_i]$ are all positive. In the former case, say $S_{j_1} \equiv S_{j_2}$ and $j_1 < j_2$.

$$\sum_{j_1+1}^{j_2} [Z_i] \equiv \sum_1^{j_1} [Z_i] + \sum_{j_2+1}^n [Z_i] \equiv 0 \pmod{n}$$

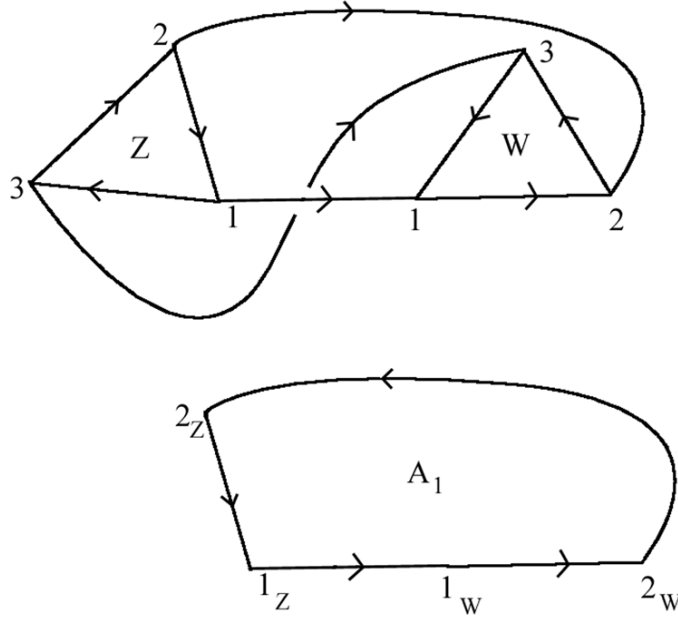
Thus, there must be some set J of the Z_i with $\sum_{i \in J} lk(L, Z_i) = kn$ with $k \neq 0$.

If J is size one, then we are done. We now induct on the size of J .

Suppose J contains r cycles. Select two of them, say Z_1 and Z_2 , and select $n+1$ vertices evenly spaced around Z_i . Since K_{η_n} is complete, each preferred vertex is adjacent to all the others. Label the vertices of Z_1 cyclically in an order agreeing with the orientation of Z_1 and label the vertices of Z_2 cyclically counter the orientation of Z_2 . We form cycles A_i by beginning at vertex i on Z_1 , taking the path along Z_1 to vertex $i+1$, then the edge from vertex $i+1$ on Z_1 to vertex $i+1$ on Z_2 , the path along Z_2 to vertex i , and then the edge from vertex i on Z_2 to vertex i on Z_1 . Note that by construction we have $n+1$ such cycles and $\sum [A_i] = [Z_1] + [Z_2] \in H_1(\mathbb{R}^3 \setminus L)$. See Figure 5.

Using the same logic as for selecting the set J we now examine the A_i . Form the partial sums $T_j := \sum_{i=1}^j [A_i]$ for $1 \leq j \leq n$. If $T_j \equiv [Z_1] + [Z_2] \pmod{n}$, then we remove Z_1 and Z_2 from J and replace them with $\cup_1^j A_i$. We have reduced the size of J by one, but have not altered the sum of the elements in $H_1(\mathbb{R}^3 \setminus L)$.

If no partial sum equals $[Z_1] + [Z_2]$ modulo n , then $T_{j_1} \equiv T_{j_2}$ for some $j_1 < j_2$. Then $\sum_{j_1+1}^{j_2} [A_i] \equiv 0$ modulo n so $\sum_1^{j_1} [A_i] + \sum_{j_2+1}^{n+1} [A_i] \equiv [Z_1] + [Z_2]$ modulo n . Let $A = \cup_{j_1+1}^{j_2} [A_i]$ and $Z'_1 = \cup_1^{j_1} [A_i] \cup \cup_{j_2+1}^{n+1} [A_i]$. Note that Z'_1 is a single cycle, as A_1 and A_{n+1} are adjacent. If $lk(A, L) \neq 0$ then we are done. If $lk(A, L) = 0$ then $lk(Z'_1, L) = lk(Z_1, L) + lk(Z_2, L)$. So, once again, we may remove Z_1 and Z_2 from J and replace them with Z'_1 .

FIGURE 5. Constructing the A_i for $n=2$

Since we may choose which cycles to pair at each step, we may pair the cycles in J with the largest number of vertices. Note that if Z_1 and Z_2 have $(n+1)m$ vertices, the A_i will contain at least $2m+2$ vertices and hence the cycle Z'_1 will contain at least $2m+2$ vertices. In the worst case, we must continue this pairing process until only a single cycle remains in J . Thus a cycle and its descendants will be used in at most $\lceil \log_2(n) \rceil$ pairings. The initial cycles Z_i contain at least $(n+1)(\lfloor \frac{n+1}{2} \rfloor)(\lceil \frac{n+1}{2} \rceil)^{\lceil \log_2(n) \rceil - 2}$ vertices, so after pairing the elements in J and producing the new cycles, J will have $\lceil \frac{n}{2} \rceil$ elements each with at least $2(\lfloor \frac{n+1}{2} \rfloor)(\lceil \frac{n+1}{2} \rceil)^{\lceil \log_2(n) \rceil - 2} + 2 \geq (n+1)(\lfloor \frac{n+1}{2} \rfloor)(\lceil \frac{n+1}{2} \rceil)^{\lceil \log_2(n) \rceil - 3} \geq (n+1)(\lfloor \frac{n+1}{2} \rfloor)(\lceil \frac{n+1}{2} \rceil)^{\lceil \log_2(\frac{n}{2}) \rceil - 2}$ vertices. The set J is now smaller, and the elements of J have sufficiently many vertices, so we may apply the induction hypothesis. \square

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